ON D-EFFICIENCY OF SOME THIRD-ORDER ROTATABLE DESIGNS

S. Huda

Indian Statistical Institute, Calcutta

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Summary

The D-efficiencies of several available third-order rotatable designs in two to eight dimensions, for regression on balls, are derived. The effect of addjng centre points to the designs on the D-efficiencies is examined. Key words : D-efficiency, Rotatable designs, Third-order.

Introduction

Comparison of designs under one or more optimality criteria has been of interest in recent years. Important contributions were made by Kiefer [13], Galil and Kiefer [5] [6] [7] [8] [9] who examined the performance under various optimality criteria of designs selected according to a particular criterion.

Rotatable designs were introduced by Box and Hunter [1] because these provide spherical variance contours for the estimated response. There is a sustantial literature on the construction of rotatable designs of low order (a d-th order design allows estimation of all the parameters in a d-th degree polynomial). Kiefer [12] showed that the D-optimum design measures for regression on balls are also rotatable. Hence, it is particularly desirable to know how the various available rotatable designs perform under the D-optimality criterion. However, no systematic investigation into the performance of available third order rotatable designs seems to have appeared in the literature. In this paper the D-efficiencies of these designs are examined.

Preliminaries : The settings under consideration are those of cubic regression on k quantitative factors $x_1, \ldots, x_k$ which are restricted to the k-ball $B_k(R)=\{(x_1,\cdots,x_k); \sum_{i=1}^{k} x_i^2 \leqslant R^2\}$. Without loss of generality it is assumed that $R=1$. A design ξ is a set of N points $\{\underset{\sim}{x}_u=(x_{1u},\ldots,x_{ku})\ (u=1,\ldots,N)\}$ in the ball, each point representing a particular combination of the levels of the factors, for fitting the cubic model

$$Y_u=\beta_0+\sum_{i=1}^{k}\beta_i x_{iu}+\sum_{i\leqslant j=1}^{k}\beta_{ij}x_{iu}x_{ju}+\sum_{i\leqslant j\leqslant l=1}^{k}\beta_{ijl}x_{iu}x_{ju}x_{lu}$$
$$+e_u(u=1,\ldots,N), \quad \ldots(1)$$

where Y_u is the response at $\underset{\sim}{x}_u$ and the e_u's are zero-mean uncorrelated errors with a constant variance σ^2.

The conditions under which a design is rotatable were derived by Box and Hunter [1]. For a third-order design these conditions are :

$$\sum_u x_{iu}^2=N\lambda_2\ (i=1,\ldots,k),$$

$$\sum_u x_{iu}^4=3\sum_u x_{iu}^2x_{ju}^2=3N\lambda_4\ (i\neq j,\ i,j=1,\ldots,k), \quad \ldots(2)$$

$$\sum_u x_{iu}^6=5\sum_u x_{iu}^4x_{ju}^2=15\sum_u x_{iu}^2x_{ju}^2x_{lu}^2=15\ N\lambda_6$$
$$(i\neq j\neq l;\ i,j,\lambda=1,\ldots,k),$$

and all other sums of powers and products upto order six are zero. A set of points satisfying (2) is called a third-order rotatable set. In order to be a non-singular third-order design the set must have points lying on two or more distinct spheres (*i.e.* satisfy $\lambda_4/\lambda_2^2>k/(k+2)$ and $\lambda_2\lambda_6/\lambda_4^2>(k+2)/(k+4)$).

The Designs: Third-order rotatable designs in 9 and higher dimensions have not been included in the study since these require too many experiments to be of any practical interest. The designs in 2 to 8 dimensions under consideration are taken from Gardiner, Grandage and Hader [4], Draper [3], Das and Narasimham [2], Herzberg [10] and Huda [11]. The designs are described in terms of the component symmetric point sets. $2^{-f}S(a_1,\ldots,a_k)$ denotes the smallest "Resolution VII" fraction of the set of all distinct permutations of $(\pm a_1,\ldots,\pm a_k)$. Similarly, $2^{-f}BIB-a(k,b,r,s,\lambda)$ denotes the smallest 'Resolution VII' fraction of the point-set generated from a $BIB(k,b,r,s,\lambda)$ (see Huda, [11].

All the designs have been scaled so that the outermost points lie on $S_k(1)$ where $S_k(R)=\{(x_1,\ldots,x_k): \sum_1^k x_i^2 = R^2\}$ is the spherical shell of radius R. The designs are described in Table 2 in which n_c denotes the number of centre points added. The D-efficiencies have been computed for $0 \leqslant n_c \leqslant 6$ since in all the cases under consideration the maximum D-efficiencies are attained within this range of n_c.

The D-efficiency: The moment matrix $\underset{\sim}{M}(\xi)$ of a third-order rotatable design is completely characterized by λ_2, λ_4 and λ_6. Further, it is readily seen that the determinant $|\underset{\sim}{M}(\xi)|$ is given by

$$|\underset{\sim}{M}(\xi)| = 2^{k^2-1}3^k \{(k+4)\lambda_2\lambda_6-(k+2)\lambda_4^2\}^k\{(k+2)\lambda_4-k\lambda_2^2\} \times \lambda_4^{(k^2+k-2)/2}\lambda_6^{k(k-1)(k+4)/6}. \qquad \ldots(3)$$

Galil and Kiefer [9] maximized (3) numerically and found that D-optimum design measure ξ^* puts mass α uniformly distributed on $S_k(\rho^*)$ $(\rho^*<1)$ and mass $(1-\alpha)$ uniformly distributed on $S_k(1)$ where for $2 \leqslant k \leqslant 8$ the values of α and ρ^* are as in Table 1 which also gives the corresponding values λ_2^*, λ_4^* and λ_6^* of λ_2, λ_4 and λ_6, respectively. The D-efficiency E_ξ of design ξ is given by

$$E_\xi = \{|M(\xi)| \,/\, |M(\xi^*)|\}^{1/p}, \qquad \ldots(5)$$

where $P=(k+3)(k+2)(k+1)/6$ is the number of parameters in (1). The value of E_ξ has been computed for several third-order rotatable designs and displayed in Table 2.

TABLE 1

Values of design parameters for D-optimum design measures

k	α	ρ^*	$100\ \lambda_2^*$	$100\ \lambda_4^*$	$100\ \lambda_6^*$
2	0.308	0.515	38.6845	8.9208	1.4536
3	0.208	0.544	28.4518	5.4014	0.7594
4	0.150	0.560	22.4260	3.6031	0.4518
5	0.113	0.569	18.4717	2.5681	0.2828
6	0.088	0.576	15.6866	1.9202	0.1907
7	0.070	0.580	13.6221	1.4888	0.1346
8	0.058	0.584	12.0223	1.1859	0.0984

TABLE 2

D-efficiency of third-order rotatable designs

Dn. No.	*k*	*Designs*	n_c	*N*	$100\lambda_2$	$100\lambda_4$	$100\lambda_6$	$100E_\xi$	*Ref.*
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	2	n equally spaced points on each of $S_k(1)$ and $S_k(\rho^*)$	0		31.631	6.690	1.061	92.989	GGH
2	2	2n and n equally spaced points on $S_k(1)$ and $S_k(\rho^*)$, respectively	0		37.754	8.626	1.402	99.847	GGH
3	2	7 equally spaced points on each of $S_k(1)$ and $S_k(\rho^*)$	1*	15	29.522	6.244	0.990	88.984	GGH
			2	16	27.677	5.853	0.929	85.103	
			3	17	26.049	5.509	0.874	81.439	
			4	18	24.602	5.203	0.825	78.018	
			5	19	23.307	4.929	0.782	74.839	
			6	20	22.141	4.683	0.743	71.890	
4	2	14 and 7 equally spaced points on $S_k(1)$ and $S_k(\rho^*)$, respectively	1*	22	36.038	8.234	1.338	97.658	GGH
			2	23	34.471	7.877	1.280	95.244	
			3	24	33.035	7.548	1.227	92.775	
			4	25	31.713	7.246	1.178	90.320	
			5	26	30.493	6.968	1.132	87.920	
			6	27	29.364	6.709	1.090	85.594	

5	2	15 and 7 equally spaced points on $S_k(1)$ and $S_k(\rho^*)$, respectively	0	22	38.310	8.803	1.433	99.976	GGH
6	2	16 and 7 equally spaced points on $S_k(1)$ and $S_k(\rho^*)$, respectively	0	23	38 819	8.963	1.461	99.998	GGH
7	3	S(1, 0, 0), S(a, a, a),	0*	32	28.099	5.041	0.663	87.235	
		S(b, 0, 0), S(c, c, 0) :	1	33	27.247	4.888	0.646	86.377	
		$a^2=0.298707$,	2	34	26.446	4.745	0.627	85.077	
		$b^2=0.401320$,	3	35	25.690	4.609	0.609	83.588	GGH
		$c^2=0.474167$	4	36	24.977	4.481	0.592	82.020	
			5	37	24.302	4.360	0 576	80.428	
			6	38	23.662	4.245	0.561	78.841	
8	3	Icosahedron : $(\pm a, \pm b, 0)$,	0	32	33.017	6.542	0.925	38.158	
		$(\pm b, 0, \pm a)$, $(0, \pm a, \pm b)$	1	33	32.016	6.343	0.898	48-287	
		and Dodecahedron :	2*	34	31 075	6.157	0.872	48.512	
		$\frac{1}{\sqrt{3}}(0, \pm c^{-1}, \pm c)$, $\frac{1}{\sqrt{3}}(\pm c, 0, \pm c^{-1})$	3	35	30.019	5.981	0.847	41.088	GGH
		$\frac{1}{\sqrt{3}}(\pm c^{-1}, \pm c, 0)$,	4	36	29.348	5.815	0.823	47.429	
		$\frac{1}{\sqrt{3}}(\pm 1, \pm 1, \pm 1)$;	5	37	28.555	5.658	0.801	46.664	
		$c^2+c^{-2}=3$, $a^2=(3+\sqrt{5})/(5+\sqrt{5})(1.012905)^2$, $b^2=2/(5+\sqrt{5})(1.012905)^2$	6	38	27.804	5.509	0.780	45.853	

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
9	3	S(1, 0, 0), S(a, a, a), S(b, c, c)	0*	50	26.028	4.248	0.512	71.423	
		2 sets of $S(\sqrt{2}a, 0, 0)$:	1	51	25.518	4 165	0.502	71.367	
		$a^2=1/(1.985406)^2$,	2	52	25.027	4.084	0.492	70.964	
		$b^2=(0.34156)^2a^2$;	3	53	24.555	4.008	0.483	70.379	GGH
		$c^2=(1.286527)^2a^2$.	4	54	24.100	3.934	0.474	69.690	
			5	55	23.662	3.862	0.465	68.939	
			6	56	23.239	3.793	0 457	68.152	
10	3	S(a, a, a), $S(2^{3/4}a, 0, 0)$,	0	44	31.411	5 927	0.800	52.197	
		S(b, 0, 0), S(c, d, d);	1	45	30.713	5.796	0.782	59.224	
		$a=1/\sqrt{3}$, $b=1.705945/\sqrt{3}$,	2*	46	30.046	5.670	0.765	59 902	
		$c=0.184388/\sqrt{3}$,	3	47	29.406	5.549	0.749	59.802	GGH
		$d=1.164944/\sqrt{3}$	4	48	28.724	5.433	0.732	59.153	
			5	49	28.206	5.322	0.718	58.588	
			6	50	27.642	5.216	0.704	57.937	
11	4	S(1, 0, 0, 0), $S(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$,	0	128	20.002	2.706	0.279	59.705	
		S(a, 0, 0, 0), S(b, b, c, c);	1	129	19.847	2,685	0 277	59.989	
		a=0.868302,	2*	130	19.694	2.664	0.275	60.046	
		b=0 600459,	3	131	19.544	2.644	0.273	59.985	GGH
		c=0.128152	4	132	19.396	2.624	0.271	59.852	
			5	133	19.250	2.604	0.269	59.672	
			6	134	19.107	2.585	0.267	59.459	

12	4	$S(1, 0, 0, 0)\ S\left(\frac{1}{\sqrt{2}},\right.$	0	96	22.426	3.397	0.391	75.776	
		$\left.\frac{1}{\sqrt{2}}, 0, 0\right)$.	1*	97	,,	3.424	0.396	75.780	
		$S(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $S(\rho, 0, 0, 0)$,	2	98	,,	3.452	0 402	75.358	
		$S\left(\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, 0, 0\right)$,	3	99	,,	3.480	0.408	74.569	D
		$S\left(\frac{\rho}{2}+\frac{\rho}{2}+\frac{\rho}{2}+\frac{\rho}{2}\right)$;	4	100	,,	3.510	0.414	73.447	
		ρ chosen to make $\lambda_2=\lambda_2^*$.	5	101	,,	3.540	0.421	71.957	
			6	102	,,	3.571	0.428	70.019	
13	4	$S(a, a, a, a)$, $S(a, a, 0, 0)$,	0*	72	20.470	3.008	0.347	82.470	
		$S(2a, 0, 0, 0)$, $S(b, b, 0, 0)$;	1	73	20.190	2.966	0.343	81.757	
		$a=\frac{1}{2}$, $b=7^{1/6}a$	2	74	19.917	2.925	0.338	81.003	
			3	75	19.652	2.886	0.333	80.224	H
			4	76	19.393	2.849	0.329	79.432	
			5	77	19.141	2.812	0.325	78.634	
			6	78	18.896	2.775	0 321	77.836	
14	4	$S\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$,	0*	72	20.351	2.994	0.347	83.051	
		$S(a, a, a, a)$, $S(2a, 0, 0, 0)$,	1	73	20.073	2.953	0.343	82.299	
		$S(b, 0, 0, 0)$, $S(c, c, c, c)$;	2	74	19.802	2.913	0.338	81.515	
		$a=1/(2\times 3^{1/3})$,	3	75	19.538	2.874	0.333	80.713	H
		$b=2\sqrt{2}\,a, c=\sqrt{2}\,a$	4	76	19.281	2.836	0.329	79.902	
			5	77	19.031	2.800	0.325	79.088	
			6	78	18.787	2.764	0 321	78.276	

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
15	4	S(1, 0, 0, 0), S(a, a, a, 0),	0*	72	20 540	2.944	0.649	72.560	
		S(b, 0, 0, 0), S(c, c, 0, 0):	1	73	20 259	2.903	0.640	72.124	
		$a^2=0.307938$,	2	74	19.985	2.864	0.631	71.527	DN
		$b^2=0.371359$,	3	75	19.719	2.826	0.623	70.986	
		$c^2=0.387977$.	4	76	19.459	2.789	0.615	70.352	
			5	77	19.206	2.752	0.607	69.697	
			6	78	18.960	2.717	0.599	69.030	
16	5	S(1, 0, 0, 0, 0),	0*	100	17.524	2.276	0.234	84.053	
		S(a, a, a, 0, 0),	1	101	17.350	2.253	0.231	83.585	
		S(b, 0, 0, 0, 0)	2	102	17.180	2.211	0.229	83.055	
		$a^2=0.307938$	3	103	17.013	1.210	0.227	82.490	DN
		$b^2=0.371359$	4	104	16.850	2.188	0.225	81.902	
			5	105	16.689	2.167	0.222	81.301	
			6	106	16.532	2.147	0.220	80.692	
17	5	$S\left(\frac{1}{\sqrt{2}}, 0, 0, 0, 0\right)$,	0*	150	17.206	2.192	0.222	82.779	
		$S\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\right)$	1	151	17.092	2.178	0.221	82.482	
		S(a, a, a, a, 0),	2	152	16.979	2.164	0.219	82.154	

		S(b, 0, 0, 0, 0),	3	153	16.868	2.150	0.218	81.802	DN
		S(c, 0, 0, 0, 0);	4	154	16.759	2.136	0.217	81.435	
		$a^2=0.218395$,	5	155	16.651	2.122	0.215	81.057	
		$b^2=0.987579$,	6	156	16.544	2.109	0.214	80.669	
		$c^2=0.428004$							
18	5	$S\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\right)$,	0*	192	15.406	1.766	0.163	66.223	
		S(a,..., a), S(0, b, b, b, b)	1	193	15.326	1.756	0.162	66.025	
		S(c, 0, 0, 0, 0).	2	194	15.247	1.747	0.161	65.813	
		S(d, 0, 0, 0, 0),	3	195	15.169	1.738	0.160	65.591	DN
		2 sets of S(e, 0, 0, 0, 0);	4	196	15.091	1.729	0.159	65.361	
		$a=1/2\sqrt{2}$, $b^2=0.198425$,	5	197	15.015	1.721	0.159	65.125	
		$c^2=0.801976$, $d^2=0.387926$,	6	198	14.939	1.712	0.158	64.884	
		$e^2=0.625$.							
19	5	S(a, a, a, a, a),	0*	124	16.656	2.048	0.202	77.941	
		S(2a, 2a, 0, 0, 0),	1	125	16.523	2.032	0.200	77.596	
		S(b, b, b, b, b), 2 sets of	2	126	16.392	2.016	0.198	77.229	
		S(2b, 0, 0, 0, 0);	3	127	19.263	2.000	0.197	76.831	Hu
		$a^2=1/8$, $b^2=3^{1/3}/8$	4	128	16.136	1.984	0.195	76.411	
			5	129	16.011	1.969	0.194	75.978	
			6	130	15.887	1.954	0.192	75.535	

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
20	5	$S\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\right)$,	0*	134	16.063	1.931	0.197	74.180	
		S(c, 0, 0, 0, 0), 2 sets of	1	135	15.944	1.917	0.185	73.830	
		S(a,..., a) and	2	136	15.827	1.903	0.184	73.460	
		S(b, 0,..., 0), each;	3	137	15.711	1.889	0.183	73.075	Hu
		$a^2=1/2^{8/3}$,	4	138	15.560	1.875	0.181	72.570	
		$c^2=0.830343$,	5	139	15.485	1.861	0.180	72.280	
		$b^2=0.446060$.	6	140	15.375	1.848	0.179	71.974	
21	5	S(a, a, 0, 0, 0). 2 sets of	0*	144	15.015	1.797	0.187	81.434	
		S(b, b, b, b, b),	1	145	14.911	1.784	0.185	80.908	
		S(c, 0, 0, 0, 0),	2	146	14.809	1.772	0.184	80.384	
		S(d, 0, 0, 0, 0) each;	3	147	14.708	1.760	0.182	79.863	Hu
		$a^2=\frac{1}{2}$; $b^2=0.157490$,	4	148	14.609	1.748	0.181	79.345	
		$c^2=0.71810$, $d^2=0.167412$	5	149	14.511	1.737	0.180	78.830	
			6	150	14.414	1.725	0.179	78.320	
22	5	$S\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, 0\right)$,	0*	182	13.598	1.508	0.144	64.982	
		S(a,..., a), S(b, 0, 0, 0, 0),	1	183	13.523	1.500	0.143	64.677	
		S(0, c, c, c, c), 2 set of	2	184	13.450	1.491	0.142	64.373	
		S(d, 0, 0, 0, 0);	3	185	13.377	1.483	0.141	64.071	Hu

		$a^2=0.189122$,	4	186	13.305	1.476	0.140	63.770	
		$b^2=0.847870$,	5	187	13.234	1.468	0.139	63.470	
		$c^2=0.111790$,	6	188	13.164	1.460	0.138	63.173	
		$d^2=0.461325$.							
23	6	S(a, a, a, 0, 0, 0),	0*	260	15.335	1.798	0.171	90.224	
		S(b, 0,..., 0), S(c, 0,..., 0),	1	261	15.276	1.791	0.170	90.074	
		S(d,..., d), S(e, 0, . . , 0);	2	262	15.218	1.784	0.169	89.896	
		$a^2=1/3$, $b^2=0.955977$,	3	263	15.160	1.778	0.169	89.696	DN
		$c^2=0.48732$, $d^2=0.132283$,	4	264	15.103	1.771	0.168	89.482	
		$e^2=0.925984$.	5	265	15.046	1.764	0.168	89.256	
			6	266	14.989	1.758	0.167	89.022	
24	6	S(2a, 2a, 0, 0, 0, 0),	0*	236	12.712	1.271	0.106	64.934	
		2 sets of S(a,..., a),	1	237	12.658	1.266	0.106	64.727	
		4 sets of S(2a, 0,..., 0):	2	238	12.605	1.261	0.105	64.516	
		$a^2=1/8$.	3	239	12.552	1.255	0.105	64.303	Hu
			4	240	12.500	1.250	0.104	64.086	
			5	241	12.448	1.245	0.104	63.871	
			6	242	12.397	1.240	0.103	63.653	
25	6	Non-central orthogonal projection of Design 29.			As Design 24				

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
26	6	S(a,..., a), S(b, b, b, 0, 0, 0),	0*	272	14.785	1.719	0.163	90.396	
		2 sets of S(c, 0,..., 0)	1	273	14.730	1.713	0.163	90.147	
		and S(d, 0,..., 0) each;	2	274	14.677	1.706	3.162	89.894	
		$b^2=1/3$, $a^2=0.132283$,	3	275	14.623	1.700	0.162	89.638	
		$c^2=0.949438$,	4	276	14.570	1.694	0.161	89.379	
		$d^2=0.32092$	5	277	14.518	1.688	0.160	89.118	
			6	278	14.465	1.682	0.160	88.856	
27	6	S(0,..., a),	0*	312	12.385	1.197	0.096	50.391	
		S(b, b, 0, 0, 0, 0),	1	313	12,346	1.194	0.096	59.257	
		S(c, 0,..., 0), 4 sets of	2	314	12.306	1.190	0.095	59.120	
		S(d, 0,..., 0); $b^2=1/2$,	3	315	12.267	1.186	0.095	58.980	Hu
		$a^2=0.146201$, $c^2=0.638298$,	4	316	12.228	1.182	0.095	58.827	
		$d^2=0.496615$.	5	317	12.190	1.179	0.095	58.693	
			6	318	12.152	1.175	0.094	58.548	
28	6	S(a, a, a, 0, 0,0),	0*	260	14.507	1.606	0.144	78.354	
		2 sets of S(b, 0,..., 0),	1	261	14.451	1.600	0.144	78.244	
		S(c. 0,..., 0),	2	262	14.396	1.593	0.143	78.103	
		S(d,.... d); $c^2=1$,	3	263	14.341	1.587	0.143	77.940	DN
		$d^2=1/8$, $a^2=0.314977$,	4	264	14.287	1.581	0.142	77.799	
		$b^2=0.629961$.	5	265	14.233	1.575	0.142	77.559	
			6	266	14.180	1.569	0.141	77.360	

29	7	S(2a, 0,.. , 0),	0*	238	12.605	1.260	0.105	82.664	
		BIB-a(7, 7, 4, 4, 2), 2 sets	1	239	12.552	1.255	0.105	82.451	
		of complementary	2	240	12.500	1.250	0.104	82.218	DN
		BIB-a (7, 7, 3, 3, 1); $a^2=\frac{1}{4}$	3	241	12.448	1.245	0.104	81.973	
			4	242	12.396	1.240	0.103	81.918	
			5	243	12.346	1.225	0.103	81.455	
			6	244	12 295	1.229	0.102	81.188	
30	7	S(a,..., a), S(2a, 0,..., 0),	0*	240	12.500	1.250	0.104	82.210	
		2 sets of S(2a, 0,..., 0),	1	241	12.448	1.245	0.104	81.973	
		$a^2=1/8$.	2	242	12.397	1.240	0.100	81.918	
			3	243	12.345	1.235	0.108	81.456	DN
			4	244	12.205	1.230	0.102	81.188	
			5	245	12.245	1.255	0.102	80.017	
			6	246	12.195	1.220	0.102	80.644	
31	7	S(1, 0, , 0),	0*	372	11.563	1.059	0.080	63.557	
		2^{-1}S(a...., a),	1	373	11.532	1.056	0.080	63.461	
		S(b, b, b, 0, 0, 0, 0),	2	374	11.501	1.052	0.080	63.357	DN
		S(c, 0,..., 0);	3	375	11.470	1.050	0.080	63.247	&
		$a^2=0.132589$,	4	376	11.440	1.047	0.079	63.131	Hu
		$b^2=3a^2$, $c^2=0.353731$	5	377	11.409	1.044	0.079	63.011	
			6	378	11.379	1.042	0 079	62.888	

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
32	8	$2^{-1}S(a, \ldots, a)$	0*	480	9.375	0.781	0 058	69.156	
		$S(2a, 2a, 0, \ldots, 0)$,	1	481	9.356	0.780	0 058	69.021	
		$S(2a, 0, \ldots, 4)$,	2	482	9.336	0.778	0.058	68.887	
		BIB-a (8, 14, 7, 4, 3, 1);	3	483	9.317	0.776	0.058	68 752	DN
		$a^2=1/8$.	4	484	9.298	0.775	0.058	68.618	
			5	485	9.278	0.773	0.058	68.485	
			6	486	9.259	0 772	0.058	68 351	
33	8	BIB-a (8, 14, 7, 4, 3, 1),	0*	480	10.669	0.937	0.071	78.182	
		$S(b, b, 0, \ldots, 0)$,	1	481	10.647	0.935	0.070	78.054	
		$2^{-1}S(e \ldots, c)$,	2	482	10.625	0.933	0.070	77.923	
		$S(d, 0, \ldots, 0)$;	3	483	10.603	0.932	0.070	77.790	DN
		$a^2=\frac{1}{4}$, $b^2=1/2\sqrt{2}$,	4	484	10.581	0.930	0.070	77.657	
		$c^2=1/8\sqrt{2}$, $d^2=1$.	5	485	10.559	0.928	0.070	77.523	
			6	486	10.538	0.926	0.070	77.388	
34	8	$2^{-1}S(a, \ \ , a)$,	0	480	12.022	1.158	0.093	86.413	
		$S(2a, 2a, 0, \ldots, 0)$.	1	481	,,	1.160	0.093	86.629	
		$2^{-1}S(b, \ldots, b)$,	2*	482	,,	1.163	0.094	86.634	
		$S(2b, 2b, 0, \ldots, 0)$;	3	483	,,	1.165	0.094	86.542	Hu
		$a^2=1/8$, b^2 chosen to give	4	484	,,	1.167	0.095	86.381	
		$\lambda_2 \quad \lambda_2^{\bullet}$.	5	485	,,	1.169	0.095	86 161	
			6	486	,,	1.172	0 095	85.886	

35	8	BIB-a (8, 14, 7, 4, 3, 1), S(2a, 0,..., 0), BIB-b (8, 14, 7, 4, 3, 1), S(2b, 0,..., 0); $a^2=\frac{1}{4}$, b^2 such that $\lambda_2=\lambda_2^*$.	As Design 34	Hu
36	8	2^{-1}S(a,..., a), S(2a, 2a, 0,..., 0), BIB-b (8, 14, 7, 4, 3, 1), S(2b, 0,..., 0); $a^2=1/8$, b^2 such that $\lambda_2=\lambda_2^*$.	As Design 34	Hu
37	8	2^{-1}S(a,..., a), S(2a, 2a, 0,..., 0), BIB-b (8, 14, 7, 4, 3, 1), S(2b, 0,..., 0); $b^2=\frac{1}{4}$, a^2 such that $\lambda_2=\lambda_2^*$.	As Design 34	Hu

Note : (*i*) *indicates the number of centre points corresponding to the highest D-efficiency.
(*ii*) GGH, D, DN, H and Hu stand for designs due to Gardiner, Grandage and Hader [4], Draper [3], Das and Narasimham [2], Herzberg [10] and Huda [11], respectively.

Comments: The four-dimensional design of Draper [3] consists of two singular third-order rotatable designs. High efficiency is achieved by placing one of these on $S_k(1)$ and the other on $S_k(\rho)$, ρ being chosen to give $\lambda_2=\lambda_2^*$; this procedure has been followed also for similar eight-dimensional designs. A possibly better alternative may have been to vary ρ so as to get $\lambda_6=\lambda_6^*$; however, this has not been tried. Of the few two-dimensional designs considered, the design consisting of 16 equally spaced points on $S_k(1)$ and 7 equally spaced points on $S_k(\rho^*)$ gives the maximum efficiency-nearly 1. Design 7 is by far the most efficient of the 3-dimensional designs. Of the five 4-dimensional designs considered the two designs due to Herzberg [10] are better than the rest. In five dimensions two of the designs due to Das and Narasimham [2] and two due to Huda [11] perform fairly well. Design 23 and Design 26 are by far better than the other 6-dimensional designs. In seven dimensions Design 29 and Design 30 have nearly equal efficiencies.

In eight dimension, the four designs due to Huda [11] have equal efficiency and are much better than the designs due to Das and Narasimham [2]. As mentioned earlier, possibly greater efficiency could be achieved by these designs if the distances of the points were varied to get $\lambda_0=\lambda_6^*$ rather than $\lambda_2=\lambda_2^*$ as done here.

In our study only D-efficiency has been investigated. There may be situations where other criteria such as A- and E-optimality are of greater interest. Efficiencies of designs under these criteria is currently being investigated and will be communicated when the study is completed. As with all theoretical studies on optimal designs, the efficiencies derived here are supposed to serve only as rough guidelines In practice the experimenter's choice of a design may depend on considerations other than theoretical efficiency.

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